STATIONARY COMBUSTION

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 6, pp. 98–100, 1966

We shall consider a semi-infinite combustion chamber into which the fuel, mixed at a temperature T_0, is supplied at a constant mass rate m from one end. It is assumed that a first-order exothermic reaction takes place in the mixture within the chamber. In the absence of heat losses through the walls, the stationary combustion problem can be reduced to the following boundary-value problem:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)-mc\frac{dT}{dx}+ha\,\Phi=0,$$

$$\frac{d}{dx}\left(\rho D\frac{da}{dx}\right)-m\frac{da}{dx}-a\Phi=0\,, \tag{1}$$

$$x=0,\ \ T=T_0,\ \ ma-\rho D\frac{da}{dx}=ma_0,$$

$$x=\infty,\ \ T=T_+,\ \ a=0\,, \tag{2}$$

where T is the temperature, a is the concentration of the fuel, k = = k(T) is the thermal conductivity, D = D(T) is the diffusion coefficient, $\rho=\rho(T)$ is the density of the mixture, c is the (constant) specific heat, h is the heat of reaction, $\Phi=\Phi(T)$ is the rate of the chemical reaction, and T_+ is the temperature of the gas reached when the fuel has been completely consumed (the unknown quantity).

A similar problem was formulated in [1] where a quantitative analysis was given of combustion in a semi-infinite chamber, and the corresponding equations were investigated for the case of similarity between the concentration and temperature fields, ($\lambda=\rho Dc/k=1$), $\rho D, c, k$ are constants and $\varphi(T)$ was an Arrhenius type function. One-dimensional combustion in an infinite gas was discussed in [2–6]. In the most general case [4] this reduces to Eq. (1) for which, instead of the boundary conditions at the end, these conditions are set at infinity, and $T_+=T_+^*$ is known. Assuming that $\Phi(T)\equiv 0$, $T_0\le T\le\varepsilon$ $(\varepsilon>T_0)$ and $\Phi(T)>0$, $\varepsilon<T\le T_+{}^*$, it was shown that when $\lambda\equiv 1$, $\lambda\equiv 0$, and $0<\lambda(T)<1$, stationary combusition is possible and unique for only one value $m=m^*$ of the mass velocity for a given gas. It was shown in [7] that it was possible to construct functions $\lambda(T)$ and $h\Phi(T)$ for which there is no solution of the corresponding boundary-value problem, at least for two values of m.

Below we shall use methods similar to those employed in [2–6] to show that in the semi-infinite combustion chamber, and for arbitrary but smooth functions k(T), D(T), $\rho(T)$, and $\Phi(T)$ satisfying the conditions

$$0\leqslant\lambda\leqslant 1\left(\lambda=\frac{\rho Dc}{k}\right),\quad \frac{d\lambda}{dT}\leqslant 0,$$

$$k,D,\rho,\frac{dk}{dT},\ \frac{d(\rho D)}{dT}\geqslant 0,\quad T_0<T\leqslant T_+\,,$$

$$\Phi,\frac{d\Phi}{dT}>0,\quad \varepsilon<T\leqslant T_+\ \ (\varepsilon\geqslant 0);\qquad \Phi\equiv 0,\quad T\leqslant\varepsilon\,, \tag{3}$$

the stationary combustion process exists and is unique for $T_0>\varepsilon$ and any rate of supply of the fuel mixture, but for $T_0<\varepsilon$ this occurs only for $m\le m^*$.

We shall use the following dimensionless variables and combinations:

$$u=\frac{T-T_0}{\gamma T_0},\quad v=\frac{ah}{\gamma cT_0},\quad \xi=\frac{x}{L},\quad \gamma=\frac{T_+-T_0}{T_0},$$

$$\gamma^*=\frac{a_0h}{cT_0}\qquad \alpha(u\gamma)=\frac{k}{mcL},\quad \beta(u\gamma)=\frac{\rho D}{mL},$$

$$\lambda(u\gamma)=\frac{\beta}{\alpha},\quad f(u\gamma)=\frac{\Phi\tau}{\rho_0}\,, \tag{4}$$

where $L=m\tau/\rho_0$ (τ is the characteristic reaction time), and γ^* is the maximum possible value of γ corresponding to combustion in an infinite gas (no heat losses through the end).

The problem defined by Eqs. (1) and (2) now becomes

$$\frac{d}{d\xi}\left(\alpha\frac{du}{d\xi}\right)-\frac{du}{d\xi}+vf=0,$$

$$\frac{d}{d\xi}\left(\beta\frac{dv}{d\xi}\right)-\frac{dv}{d\xi}-vf=0; \tag{5}$$

$$\xi=\infty,\quad u=1,\quad v=0;$$

$$\xi=0,\quad u=0,\quad v-\alpha\frac{dv}{d\xi}=\frac{\gamma^*}{\gamma} \tag{6}$$

From Eq. (5) and the boundary conditions at infinity we have

$$\alpha\frac{du}{d\xi}+\beta\frac{dv}{d\xi}-v-u+1=0\,. \tag{7}$$

If we take u as the independent variable, and v and $p=\alpha du/d\xi$ as the unknown functions and, moreover, if we replace the second equation in Eq. (5) by the integral of Eq. (7), then instead of Eqs. (5) and (6) we have

$$\frac{dp}{du}=1-\frac{v\varphi(u\gamma)}{p}\qquad(\varphi=f\alpha)\,, \tag{8}$$

$$\frac{dv}{du}=\frac{v+u-1-p}{\lambda(u\gamma)\,p},\quad \lambda\neq 0;$$

$$v=p-u+1,\qquad \lambda=0\,, \tag{9}$$

$$u=1,\ \ p=0,\ \ v=0\,, \tag{10}$$

$$u=0,\ \ p=\gamma^*/\gamma-1\,, \tag{11}$$

where Eq. (11) was obtained from the second condition in Eq. (6) using Eq. (7).

The point u = 1, p = 0, v = 0 is a singularity. Three pairs of integral curves pass through it, two of which give p < 0 which is in conflict with Eq. (11). The remaining curves have slopes

$$\left(\frac{dp}{du}\right)_{u=1}=k_1=\frac{1-\sqrt{1-\varphi(\gamma)\lambda(\gamma)}}{2\lambda(\gamma)}\,,$$

$$\left(\frac{dv}{du}\right)_{u=1}=k_2=\frac{k_1(1-k_1)}{\varphi(\gamma)},\quad \lambda(\gamma)\neq 0$$

$$k_1=-\varphi(\gamma),\ \ k_2=k_1-1,\ \ \lambda(\gamma)=0\,. \tag{12}$$

These curves define the unique solution $p(u,\gamma)$, $v(u,\gamma)$ of the problem specified by Eqs. (8)–(10) for any $0\le\gamma\le\gamma^*$. It remains to determine the existence of values $\gamma=\gamma^\circ$ for which Eq. (11) is satisfied.

Suppose that $T_0>\varepsilon$. We then have from Eqs. (3) and (4)

$$\varphi>0,\quad \lambda_\gamma\leqslant 0,\quad 0\leqslant u\leqslant 1;$$

$$\varphi_\gamma>0,\ (\varphi\lambda)_\gamma=(f\beta)_\gamma>0,\quad 0<u\leqslant 1\,, \tag{13}$$

where the subscript γ indicates a partial derivative with respect to γ. We shall show that, in this case,

$$0<p<\infty,\quad 0<v<\infty,\quad 0\leqslant u<1\,. \tag{14}$$

If p and v, which according to Eq. (12) are positive in the left-hand neighborhood of u = 1, were to intersect the u-axis for $0\le u<1$, then the point u° would lie in this interval at which either p = 0, $v\ge 0$, $dp/du\ge 0$, or v = 0, p > 0, $dv/du\ge 0$. It follows from Eqs. (8) and (9) that such a point cannot exist and, therefore, p and v are finite.

Let us establish the following preliminary inequalities:

$$z_1=v+u-1\geqslant 0,\qquad z_2=v+u-1-p\leqslant 0\,. \tag{15}$$

Differentiating the expressions for z_i ($i = 1, 2$) term by term and using Eqs. (8)-(10), we obtain

$$\frac{dz_i}{du} = \frac{z_i}{\lambda p} + A_i, \qquad \lambda \neq 0 \quad \left(A_1 = \frac{\lambda - 1}{\lambda}, \ A_2 = \frac{v\varphi}{p}\right), \quad (16)$$

$$z_i = (2 - i)\, p, \ \lambda = 0; \quad z_i = 0, \ u = 1. \qquad (17)$$

If we regard Eq. (16) as the equation for $z_i(u, \gamma)$ with known p and v, and Eq. (17) as the boundary conditions for it, then in the case of λ equal and not equal to zero in the range (u, 1), the solution of the problem specified by Eqs. (16) and (17) will be

$$z_i(u, \gamma) =$$

$$= (2 - i)\, p(u_0, \gamma) \exp\left(-\int_u^{u_0} \frac{du}{\lambda p}\right) - \int_u^{u_0} A_i \exp\left(-\int_u^{u_1} \frac{du_2}{\lambda p}\right) du_1,$$

$$u_0 = 1, \quad \text{if} \quad \lambda(u_1\gamma) \neq 0, \ u \leqslant u_1 \leqslant 1,$$

$$u_0 = u_1^0, \quad \text{if} \quad \lambda(u_1\gamma) \neq 0, \ u \leqslant u_1 < u_1^0, \ \lambda(u_1^0\gamma) = 0. \ (18)$$

The validity of Eq. (18) follows from Eqs. (3), (14), and (15).

We shall now show that

$$p_\gamma > 0, \ v_\gamma \geqslant 0, \ 0 < u < 1; \ p_\gamma, \ v_\gamma \geqslant 0, \ u = 0. \qquad (19)$$

These inequalities are satisfied in the neighborhood of u = 1 since $p_\gamma = v_\gamma = 0$, $dp_\gamma/du = dk_1/d\gamma < 0$, $dv_\gamma/du \leq 0$, u = 1 in accordance with Eqs. (10), (12), (13), and (3). If the inequalities given by (19) are subsequently violated, then we could find a point $0 < u^\circ < 1$, where either $p_\gamma = 0$, $v_\gamma \geq 0$, $dp_\gamma/du \geq 0$, or $v_\gamma < 0$, $p_\gamma \geq 0$, $dv_\gamma/du \geq 0$.

On the other hand, from Eq. (8) and (9) we have

$$\frac{dp_\gamma}{du} = -\frac{\varphi}{p} v_\gamma + \frac{v\varphi}{p^2} p_\gamma - \frac{v\varphi_\gamma}{p},$$

$$\frac{dv_\gamma}{du} = \frac{v_\gamma}{\lambda p} - \frac{z_1}{\lambda p^2} p_\gamma - \frac{z_2}{\lambda^2 p} \lambda_\gamma, \quad \lambda \neq 0;$$

$$v_\gamma = p_\gamma, \quad \lambda = 0 \ , \qquad (20)$$

and hence using Eq. (3), (13), (14), and (15), we readily see that the above point u° cannot exist.

From the inequalities given by (19) and (14) it follows that $p(0, \gamma)$ does not decrease as γ varies from zero to γ^*, and remains finite and positive everywhere. At the same time, the required value of p of Eq. (11) decreases monotonically from infinity to zero. There will be, therefore, one and only one value of γ° for which these quantities will be equal. Therefore, in the case of the problem defined by Eqs. (8)-(11) there is a unique solution $p(u, \gamma^\circ)$, $v(u, \gamma^\circ)$ for any m. It is readily seen that the functions $u(\xi)$ and $v(\xi)$ are the solution of the problem of Eqs. (5) and (6) and are also unique.

Suppose now that $T_0 < \varepsilon$, i.e.,

$$f(u\gamma) \equiv 0, \qquad 0 \leqslant u \leqslant \delta(\gamma) \qquad \left(\delta = \frac{\varepsilon - T_0}{\gamma T_0}\right).$$

The inequalities given by (13), (14), and (19) are then satisfied for $\delta < u < 1$ and $(p_\gamma)_{u=\delta} \geq 0$. Moreover, it can be shown in a similar way that $\partial p/\partial m = p_m < 0$, $v_m \leq 0$, $\delta \leq u < 1$. According to Eq. (8), $dp_\gamma/du = dp_m/du = 0$, $0 \leq u \leq \delta$ and therefore

$$p_\gamma \geqslant 0, \ p_m < 0, \ u = 0 \ . \qquad (21)$$

It follows from the theory of combustion in an infinite gas [2-6] that the value $\gamma^\circ = \gamma^*$ will ensure a solution of the problem defined by Eqs. (8)-(11) for $m = m^*$. According to Eqs. (21) and (11), the value of γ° will be unique for $m = m^*$. For $m < m^*$ the quantity $p(0, \gamma^*)$ will be positive in view of Eq. (21), and negative for $m > m^*$. Therefore, according to Eqs. (21) and (11), as γ decreases for $m < m^*$ there will be a unique value $\gamma^\circ < \gamma^*$ for which Eq. (11) will be again satisfied, but no such value will exist for $m > m^*$. We note that when $m = m^*$ there is no heat loss through the end, since $p(0, \gamma^*) = 0$ and, consequently, the flame will be at infinity. With decreasing m the quantity $p(0, \gamma^\circ)$ will increase, i.e., the flame will increasingly approach the end.

We note that both for one-dimensional combustion in an infinite gas and in the present case it is convenient to replace m by the Peclet number $P_T = 1/\alpha$. The results are quite similar. Thus, in the combustion chamber which we are considering, and for the above assumptions, the stationary combustion state exists and is unique for any P_T when $T_0 > \varepsilon$, and only for $P_T \leq P_T^*$ when $T_0 < \varepsilon$ (P_T^* is the Peclet number for which unique stationary combustion in an infinite gas is possible).

In conclusion, I should like to thank Yu. S. Ryazantsev for valuable suggestions.

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14 July 1966 Moscow